University of Chester

Department of Mathematics

Applied Mathematics Group
Research Report

2007 Series

CONCERNING PERIODIC SOLUTIONS TO NON-LINEAR DISCRETE VOLTERRA EQUATIONS WITH FINITE MEMORY

Christopher T H Baker Yihong Song

Report 2007:1

Concerning Periodic Solutions of Non-linear Discrete Volterra Equations with Finite Memory

Christopher T H Baker [a,1] Yihong Song [b,2]

[a] *Department of Mathematics, University of Chester, CH1 4BJ, Emeritus professor, School of Mathematics, University of Manchester, UK*

[b] *Department of Mathematics, Suzhou University, Jiangsu 215006, P. R. China*

Abstract

In this paper we discuss the existence of periodic solutions of *discrete* (and *discretized*) non-linear Volterra equations with finite memory. The literature contains a number of results on periodic solutions of non-linear Volterra *integral equations* with finite memory, of a type that arises in biomathematics. The "summation" equations studied here can arise as discrete models in their own right but are (as we demonstrate) of a type that arise from the discretization of such integral equations. Our main results are in two parts: (*i*) results for discrete equations and (*ii*) consequences for quadrature methods applied to integral equations. The first set of results are obtained using a variety of fixed point theorems. The second set of results address the preservation of properties of integral equations on discretizing them. An expository style is adopted and examples are given to illustrate the discussion.

Key words:
Periodic solutions, discrete equations, finite memory, fixed point theorems, quadrature, simulation.
AMS Subject Classification: 65Q05, 37M05, 39A12, 47B34, 47B60

1 Introduction

The equation

$$\mathsf{x}(t) = \int_{t-\tau}^{t} \mathsf{k}(t,s)\mathsf{f}(s,\mathsf{x}(s))\mathrm{d}s, \quad \text{for } t \in \mathbb{R} \text{ with } \tau > 0,\ \mathsf{x}(t) \in \mathbb{R}, \tag{1.1}$$

is an integral equation of Volterra type, studied [9] as a model of certain epidemic problems. We study periodic solutions of an analogous *discrete* system

$$x(n) = \sum_{j=n-N}^{n} k(n,j)f(j,x(j)), \quad N \in \mathbb{N}, \quad x(n) \in \mathbb{R}, \tag{1.2}$$

[1] E-mail: `cthbaker@na-net.ornl.gov`.
[2] E-mail: `yihongsong@hotmail.com` Supported in part by NNSF of China (No. 10471102).

given N, under certain conditions on $\{k(n,j)\}$ and $\{f(n,x(n))\}$ (see below). Here, $\mathbb{R} = (-\infty,\infty)$, $\mathbb{R}_+ = [0,\infty)$, we write $\mathbb{Z}$ to denote the integers, $\mathbb{Z}_+$ the set of non-negative integers, $\mathbb{N}$ the positive integers $\mathbb{Z}_+ \setminus \{0\}$. The set of quotients of positive integers we denote by $\mathbb{Q}_+$.

For each integer n, eqn (1.2) is (if $k(n,n) \neq 0$) is an *implicit* equation for $x(n)$. One may consider (1.2) for $n \in \mathbb{Z}$, for $n \geq n_0$ ($n \in \mathbb{Z}$), or for $n \in \mathbb{Z}_+$, for example. A particular solution $\{x(\varphi;n)\}_{n\geq 0}$ of (1.1) for $n \in \mathbb{Z}_+$ corresponds to a choice

$$x(\varphi;n) := \varphi(n) \text{ for } n \in \{-N,\dots,-1\}; \tag{1.3}$$

$x(\varphi;n)$ is an analogue of a solution $\mathsf{x}(\boldsymbol{\phi};t)$ of (1.1) for $t \geq 0$, wherein $\mathsf{x}(\boldsymbol{\phi};t) = \boldsymbol{\phi}(t)$ for $t \in [-\tau,0]$. There is no reason, without imposing assumptions on k and f, to suppose that eqn (1.2) has a solution $x(n)$ given $\{x(j)\}_{j<n}$.

Eqn (1.2) is a discrete analogue of (1.1), and (1.2) arises in the numerical analysis of (1.1) (see below, and [1,4,8]), as well as in discrete models. Given the terminology "integral equation" for (1.1), it seems appropriate to term (1.2) a "summation equation" (of Volterra type, with a finite memory). Eqn (1.2) has also been called a recurrence relation or a difference equation.

Example 1.1 *Suppose that $\tau > 0$ and $\mathfrak{h} > 0$ is chosen with $N_\tau \in \mathbb{N}$ and $\mathfrak{h} = \tau/N_\tau$, and suppose that $\vartheta \in [0,1]$. Then the* "composite" *or* "repeated" *version (§4.1) of the ϑ-rule applied to (1.1) yields the form (1.2) with*

$$k(n,j) = \mathfrak{h}\mathsf{k}(n\mathfrak{h}, j\mathfrak{h}) \text{ for } j \notin \{n-N_\tau, n\}, \tag{1.4a}$$

$$k(n,n-N_\tau) = (1-\vartheta)\mathfrak{h} \times \mathsf{k}(n\mathfrak{h},(n-N_\tau)\mathfrak{h}), \quad k(n,n) = \vartheta\mathfrak{h} \times \mathsf{k}(n\mathfrak{h},n\mathfrak{h})$$

and

$$f(n,v) = \mathsf{f}(n\,\mathfrak{h}, v). \tag{1.4b}$$

By the opening assumptions, $\mathfrak{h} \in \mathfrak{H}_\tau$ where we define (for any $\sigma \in \mathbb{R}_+$)

$$\mathfrak{H}_\sigma := \{\sigma/N_\sigma \text{ for some } N_\sigma \in \mathbb{N}\}. \tag{1.5}$$

In the case $\vartheta = 0$, we obtain equations $x(n) = \sum_{j=n-N_\tau}^{n-1} \mathfrak{h}\mathsf{k}(n\mathfrak{h},j\mathfrak{h})\mathsf{f}(j\mathfrak{h},x(j))$. These relations are explicit *and are special cases (for $N = N_\tau$) of the form*

$$x(n) = \sum_{j=n-N}^{n-1} k(n,j) f(j,x(j)), \tag{1.6}$$

However, the cases $\vartheta = \frac{1}{2}$ and $\vartheta = 1$ yield implicit recurrence relations and it is known that these relations display, in many cases, better stability properties than $\vartheta = 0$. Under Lipschitz conditions, the convergence rates of $x(\varphi;n)$ to $\mathsf{x}(\boldsymbol{\phi};t)$, as $\mathfrak{h} \searrow 0$, where $\varphi(j) = \boldsymbol{\phi}(j\mathfrak{h})$ for $j \in [-\tau,0]$ (and $n\mathfrak{h} = t$, $n \to \infty$), are optimized by the choice $\vartheta = \frac{1}{2}$.

It appears that little work has been done on questions of periodic solutions of the *implicit*, *non-linear*, *finite-memory* discrete system [3] (1.2). This motivates

[3] For periodic solutions of *explicit* discrete systems, see, *e.g.*, [10,12]. Periodic solutions of (1.6) with $f(j,x(j)) = x(j)$ were discussed in [11]. For periodic solutions of *implicit* discrete systems with *unbounded memory*, see [7,18].

us to investigate periodic solutions of (1.2), and to parallel some results for (1.1) stated in [2]. We establish existence results for periodic solutions of (1.2), via a variety of fixed point theorems (*cf.* [3]). In §2, we give some basic results and recall several fixed point theorems, and give our main results in §3. In §4, we discuss quadrature methods for (1.1) and demonstrate the application of the general results of §3. The present paper developed out of Song [17].

Example 1.2 *If $k(t,s) \equiv 1$, then (1.1) reduces to the integral equation*

$$\mathsf{x}(t) = \int_{t-\tau}^{t} \mathsf{f}(s, \mathsf{x}(s))\mathrm{d}s. \tag{1.7}$$

Examples studied in the literature also include those of the form

$$\mathsf{x}(t) = \int_{t-\tau}^{t} \mathsf{k}(t,s)\mathsf{g}(s)\{\mathsf{x}(s)\}^{\gamma}\ \mathrm{d}s \tag{1.8a}$$

and, more generally,

$$\mathsf{x}(t) = \int_{t-\tau}^{t} \mathsf{k}(t,s)\mathsf{g}(s)[\{\mathsf{x}(s)\}^{\alpha} + \{\mathsf{x}(s)\}^{\beta}]\ \mathrm{d}s \tag{1.8b}$$

subject to certain conditions [2]. We shall give some results for discrete versions of (1.8a) and (1.8b) in the present paper.

Remark 1.1 *Eqn (1.7) was discussed, in [9], as a model for the spread of certain infectious diseases with periodic contact rate that varies seasonally, where $\tau > 0$ is the length of time an individual remains infectious and it is assumed that there exists $\varpi > 0$ with $\mathsf{f}(t+\varpi, u) = \mathsf{f}(t,u)$ for all $t \in \mathbb{R}$ and $u \in \mathbb{R}_+$ (see [9] for details). Using Krasnosel′skiĭ's fixed point theorem in a cone [14, p.137], Cooke and Kaplan [9] established the existence, if τ is sufficiently large, of a nontrivial periodic non-negative solution to (1.7) with period ϖ. Later, Leggett and Williams [15] generalized the results in [9].*

2 Preliminaries to the fixed point analysis

For basic functional analysis see, *e.g.*, [16]. Suppose that X is a linear space; if $\|\cdot\|$ is a norm on X we denote by $X_{\|\cdot\|}$ the corresponding normed linear space and if $X_{\|\cdot\|}$ is a Banach space we denote it by $\mathcal{X}$ or $\{X; \|\cdot\|\}$. The closure of a set S is denoted $\overline{\mathsf{S}}$, and the boundary $\partial\mathsf{S}$ of an open set S is $\partial\mathsf{S} := \overline{\mathsf{S}} \setminus \mathsf{S}$. For the basic properties of *convex sets* and *cones* [4] (needed below) see [13,14]. Denote by $\ell(\mathbb{Z})$ the linear space whose elements are sequences with real $x(n)$:

$$\ell(\mathbb{Z}) := \{x \mid x = \{x(n)\}_{n\in\mathbb{Z}},\ x(n) \in \mathbb{R}\} \tag{2.1}$$

(x or $\{x(n)\}_{n\in\mathbb{Z}}$ denotes $\{\ldots, x(-2), x(-1), x(0), x(1), x(2), \ldots\}$, and we have the obvious definitions of addition and scalar multiplication).

[4] $\mathsf{C} \subset X$ is a cone in $X_{\|\cdot\|}$ if it is a closed convex set and (i) for any non-zero $v \in \mathsf{C}$ and any $\lambda \geq 0$, we have $\lambda v \in \mathsf{C}$, and (ii) if $v \in X$ is non-zero then at least one of the pair $\{v, -v\}$ does not lie in C.

Definition 2.1 *Let $\omega \in \mathbb{N}$ be a given positive integer. Then $x = \{x(n)\}_{n\in\mathbb{Z}}$ is an* ω-periodic sequence *if $x(n+\omega) = x(n)$ for all $n \in \mathbb{Z}$. $A_\omega = A_\omega(\mathbb{Z})$ denotes the (finite-dimensional) subspace of $\ell(\mathbb{Z})$ consisting of all ω-periodic sequences. We define the norm $|\cdot|_\omega$ on $A_\omega(\mathbb{Z})$ by setting $|x|_\omega = \sup_{n\in\mathbb{Z}} |x(n)| = \max_{1\le n\le\omega} |x(n)|$ for $x = \{x(n)\}_{n\in\mathbb{Z}} \in A_\omega(\mathbb{Z})$. A solution x of (1.2) is called periodic if it satisfies (1.2) for $n \in \mathbb{Z}$ and $x \in A_\omega$ for some integer $\omega \in \mathbb{N}$.*

Obviously, an ω-periodic sequence is $\omega_\star$-periodic where $\omega_\star/\omega \in \mathbb{N}$. We next state, as lemmata, two basic results (applicable for given $\omega \in \mathbb{N}$).

Lemma 2.2 $\mathcal{A}_\omega = \{A_\omega(\mathbb{Z}); |\cdot|_\omega\}$ *is an ω-dimensional Banach space.*

Lemma 2.3 *Let $\mathsf{D} \subseteq \mathcal{A}_\omega$. If an operator $T : \mathsf{D} \to \mathcal{A}_\omega$ is continuous and maps bounded sets into bounded sets, then T is completely continuous (compact).*

For Lemma 2.3, recall that $T : \mathsf{D} \to \mathcal{A}_\omega$ is *completely continuous*, or *compact*, if, and only if, for any bounded sequence $\{x_n\}$ in D, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ is convergent in $\mathcal{A}_\omega$. By the general Heine-Borel theorem, every bounded set in $\mathcal{A}_\omega$ contains a convergent sequence.

The existence of one or more periodic solutions of (1.2) will be established via fixed point theories. We associate with (1.2) an operator T on $\ell(\mathbb{Z})$ (or a subspace), with $(Tx)(n) := \sum_{j=n-N}^{n} k(n,j) f(j, x(j))$, and we identify a solution of (1.2) as a fixed point of T. We will use (citing, and paraphrasing, [2]) either *Krasnosel′skiĭ's fixed point theorem* (stated here as Theorem 2.4 – see [2, Theorem 2.1.1]), the *"non-linear alternative"* (Theorem 2.5– see [2, Theorem 1.2.1]) or the *Leggett-Williams fixed point theorem* (Theorem 2.6– see [2, Theorem 4.3.1]). These theorems were used in [2] to discuss (1.1).

Theorem 2.4 (Krasnosel′skiĭ) *Let $\mathcal{X} \equiv \{X; \|\cdot\|\}$ be a Banach space and let $\mathsf{C} \subset \mathcal{X}$ be a cone in $\mathcal{X}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{X}$ with $0 \in \Omega_1$, $\overline{\Omega}_1 \subseteq \Omega_2$, and let $T : \mathsf{C} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathsf{C}$ be a completely continuous operator such that either (i) $\|Tu\| \le \|u\|$, $u \in \mathsf{C} \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|, u \in \mathsf{C} \cap \partial\Omega_2$ or (ii) $\|Tu\| \ge \|u\|$, $u \in \mathsf{C} \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|, u \in \mathsf{C} \cap \partial\Omega_2$ is true. Then T has a fixed point in $\mathsf{C} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.*

Theorem 2.5 (Non-Linear alternative) *Suppose that K is a convex subset of a normed linear space $X_{\|\cdot\|}$, and let U be an open subset of K, with $p^\star \in \mathsf{U}$. Then every compact, continuous map $T_\natural : \overline{\mathsf{U}} \to \mathsf{K}$ has at least one of the following two properties: (i) $T_\natural$ has a fixed point in $\overline{\mathsf{U}}$; (ii) there is an $x_\lambda \in \partial\mathsf{U}$ with $x_\lambda = (1-\lambda)p^\star + \lambda T_\natural x_\lambda$ for some $0 < \lambda < 1$.*

Note that (i) and (ii) are not (as sometimes implied by the description *alternative*) mutually exclusive, but if conclusion (ii) is shown to be false then conclusion (i) holds. In applications, K can be a cone.

Theorem 2.6 (Leggett-Williams) *Let $X_{\|\cdot\|}$ define a Banach space $\mathcal{X}$, $\mathsf{C} \subset X$ a cone in $\mathcal{X}$, $r_1, r_2 \in (0,\infty)$, $r_1 \ne r_2$ with $R = \max\{r_1, r_2\}$ and $r = \min\{r_1, r_2\}$. Define $C_\eta = \{x \in \mathsf{C} : \|x\| < \eta\}$ ($\partial C_\eta = \{x \in \mathsf{C} \,\big|\, \|x\| = \eta\}$, $\overline{C}_\eta = \{x \in C \,\big|\, \|x\| \le \eta\}$). Let $T : \overline{C}_R \to \mathsf{C}$ be a continuous, compact map* *such that (i) there exists* $u_0 \in \mathsf{C}\backslash\{0\}$ *with* $Tu \nleqslant u$ *for* $u \in \partial C_r \cap C(u_0)$ *where* $C(u_0) = \{u \in C \,\big|\, \exists \lambda > 0$ *with* $u \geqslant \lambda u_0\}$*; and (ii)* $\|Tu\| \le \|u\|$ *for* $u \in \partial C_R$*. Then* T *has at least one fixed point* $x \in \mathsf{C}$ *with* $r \le \|x\| \le R$*.*

3 Main results on discrete equations

The authors' main results will be stated as PROPOSITIONS. Associated with (1.2), we define (for suitable k, f) an operator T on $\ell(\mathbb{Z})$ by

$$(Tx)(n) = \sum_{j=n-N}^{n} k(n,j) f(j, x(j)), \quad n \in \mathbb{Z}. \tag{3.1}$$

A fixed point of T is a solution of (1.2). We shall rely on assumptions that parallel assumptions made in discussions [2] of the continuous case (1.1). We shall also make additional varying assumptions from amongst the following together with additional conditions. We have

$$\omega \in \mathbb{N}. \tag{3.2}$$

Throughout, one may replace ω by $\omega_\star = \widehat{\jmath} \times \omega$ with given $\widehat{\jmath} \in \mathbb{N}$.

ASSUMPTION 1

$$k : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}_+ \text{ and } k(n+\omega, j+\omega) = k(n,j) \text{ for every } n, j \in \mathbb{Z}. \tag{3.3}$$

ASSUMPTION 2 *(a)* $$f : \mathbb{Z} \times \mathbb{R}_+ \to \mathbb{R}_+ \tag{3.4a}$$

ASSUMPTION 2 *(b)* $$f(n,\cdot) \text{ is continuous on } \mathbb{R}_+ \text{ for each } n \in \mathbb{Z}. \tag{3.4b}$$

ASSUMPTION 2 *(c)* $$f(n+\omega, u) = f(n,u) \text{ for all } n \in \mathbb{Z} \text{ and } u \in \mathbb{R}_+. \tag{3.4c}$$

ASSUMPTION 3 *(a) The function* ψ *is a non-decreasing continuous map*

$$\psi : \mathbb{R}_+ \to \mathbb{R}_+; \;\; \psi(u_1) \ge \psi(u_2) \text{ if } u_1 \ge u_2 \in \mathbb{R}_+. \tag{3.5a}$$

ASSUMPTION 3 *(b) The functions* f, ψ *in (3.5a), and* $q \in \mathcal{A}_\omega$ *satisfy*

$$f(n,u) \le q(n)\psi(u) \text{ for all } u \in \mathbb{R}_+ \text{ and } n \in \mathbb{Z}. \tag{3.5b}$$

ASSUMPTION 3 *(c) There exists a constant* $a_0 \in (0,1)$*, such that the functions* f, ψ *in (3.5a), and* $q \in \mathcal{A}_\omega$ *satisfy*

$$a_0 q(n)\psi(u) \le f(n,u) \text{ for all } u \in \mathbb{R}_+ \text{ and } n \in \mathbb{Z}. \tag{3.5c}$$

ASSUMPTION 3 *(d) There exists a function* $\xi : (0,1) \to \mathbb{R}_+\backslash\{0\}$ *such that, for* ψ *in (3.5a),*

$$\psi(\mu v) \ge \xi(\mu)\psi(v), \text{ for any } 0 < \mu < 1,\ v \ge 0. \tag{3.6}$$

ASSUMPTION 4 *With given* f, ψ, q*, there exists a continuous function* $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ *with*

$$\chi(u)q(n) \le f(n,u) \le q(n)\psi(u) \text{ for all } u \in \mathbb{R}_+ \text{ and } n \in \mathbb{Z}. \tag{3.7}$$

(The second inequality in (3.7) arises in (3.5b).) In Proposition 3.5 we require (3.7) with $\chi(u)/u$ nonincreasing for u in an interval $(0,r)$. Some further assumptions are stated, later, in terms of $\kappa_{\max}$, $\kappa_{\min}$:

Definition 3.1 *$\kappa_{\max,\min}(q_\natural)$ are defined for any $q_\natural \in A_\omega$ by*

$$\kappa_{\max}(q_\natural) := \max_{1\le n\le\omega} \sum_{j=n-N}^{n} k(n,j)q_\natural(j), \;\; \kappa_{\min}(q_\natural) := \min_{1\le n\le\omega} \sum_{j=n-N}^{n} k(n,j)q_\natural(j). \quad (3.8)$$

The following assumptions relate to the integral equation (1.1).

ASSUMPTION 5 *We have $\varpi \in \mathbb{R}_+$ and*

$$\mathsf{k} \in C(\mathbb{R}\times\mathbb{R} \to \mathbb{R}_+) \text{ and, for every } t,s \in \mathbb{R}, \mathsf{k}(t+\varpi, s+\varpi) = \mathsf{k}(t,s). \quad (3.9a)$$

Further,

$$\mathsf{f} : C(\mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}_+), \text{ and } \mathsf{f}(t+\varpi, u) = \mathsf{f}(t,u) \quad (3.9b)$$

PROPOSITION 3.1 *Suppose that* ASSUMPTION *5 applies. When $\mathfrak{h} > 0$, when*

$$\mathfrak{h}W_\ell \in \mathbb{R}_+, \quad \text{for } \ell \in \{n-N, n-N+1, \ldots, n\}, \; n \in \mathbb{Z} \quad (3.10)$$

and k and when k, f and f are related by

$$k(n,j) = \mathfrak{h}W_{n-j}\mathsf{k}(n\mathfrak{h}, j\mathfrak{h}), \; j \in \{n-N, n-N+1, \ldots, n\}, \; n \in \mathbb{Z} \quad (3.11)$$

$$f(n,u) = \mathsf{f}(n\mathfrak{h}, u), \; n \in \mathbb{Z}, \; u \in \mathbb{R}, \quad (3.12)$$

then ASSUMPTIONS *1 and 2 are satisfied with $\varpi = \mathfrak{h} \times \omega$ where $\omega \in \mathbb{N}$. Suppose, additionally, that*

$$\mathsf{q} \in C(\mathbb{R} \to \mathbb{R}) \text{ and } \mathsf{q}(t) = \mathsf{q}(t+\varpi) \text{ for } t \in \mathbb{R}. \quad (3.13a)$$

Then if $q(n) = \mathsf{q}(nh)$ (for $n \in \mathbb{Z}$) and $\varpi = \mathfrak{h} \times \omega$, it follows that $q \in \mathcal{A}_\omega$. Relations (3.5b) or (3.5c) follow if, respectively,

$$\mathsf{f}(t,u) \le q(t)\psi(u) \text{ for all } u \in \mathbb{R}_+ \text{ and } t \in \mathbb{R}, \quad (3.13b)$$

or

$$a_0\mathsf{q}(t)\psi(u) \le \mathsf{f}(t,u) \text{ for all } u \in \mathbb{R}_+ \text{ and } t \in \mathbb{R}. \quad (3.13c)$$

Remark 3.1 *Conditions on k, f and on q, in the literature on (1.1), include the assumptions on k, f and on q of Proposition 3.1. Now, for $\varpi \in \mathbb{R}_+$, $\mathfrak{H}_\varpi \subset \mathbb{R}$ is defined as $\{\varpi/N_\varpi \mid N_\varpi \in \mathbb{N}\}$. In view of Proposition 3.1, we later suppose that $h \in \mathfrak{H}_\varpi$. Referring to Example 1.1, where $h = \tau/N_\tau$, we also ask that $h \in \mathfrak{H}_\tau$. This restricts τ and ϖ (we need $\tau = \{N_\tau/N_\varpi\}\varpi$). This restriction will be overcome; see Proposition 4.1 et seq.*

3.1 Positive periodic solutions via Krasnosel′skiĭ's fixed point theorem

First, we use Krasnosel′skiĭ's fixed point theorem to establish a result for (1.2). In Krasnosel′skiĭ's theorem, take $\mathcal{X}$ to be $\mathcal{A}_\omega \equiv \{A_\omega, |\cdot|_\omega\}$, and $\mathsf{C} = \{x \in$

$\mathcal{A}_\omega \mid x(n) \geq M|x|_\omega$ for $n \in \mathbb{Z}\}$ with M from (3.14b): $x \in C$ when $x(n) \geq M|x|_\omega$ for $1 \leq n \leq \omega$. It is readily shown that C is a cone in $\mathcal{A}_\omega$. Let $T : C \to \ell(\mathbb{Z})$ be defined by (3.1). Then, $T : C \to \mathcal{A}_\omega$. Indeed, for any $x \in C \subset \mathcal{A}_\omega$, it follows from (3.3) and (3.4a) that

$$(Tx)(n+\omega) = \sum_{j=n+\omega-N}^{n+\omega} k(n+\omega, j) f(j, x(j)) =$$

$$\sum_{j=n-N}^{n} k(n+\omega, j+\omega) f(j+\omega, x(j+\omega)) = \sum_{j=n-N}^{n} k(n,j) f(j, x(j)) = (Tx)(n),$$

so $Tx \in \mathcal{A}_\omega$. Next we require that $T : C \to \mathcal{A}_\omega$ is continuous and compact.

Continuity is readily established; consider the compactness of T. Let Ω be a bounded set in C (there exists $r > 0$ with $|x|_\omega \leq r$ for all $x = \{x(n)\}_{n\in\mathbb{Z}} \in \Omega$). Since $f(j,\cdot)$ is continuous on $\mathbb{R}_+$ for each $j \in \{1,\ldots,\omega\}$, there exists $M_r > 0$ such that $|f(j,u)| \leq M_r$ for all $u \in [0,r]$ and $j \in \{1,\ldots,\omega\}$. For $n \in \{1,\ldots,\omega\}$ and $x = \{x(n)\}_{n\in\mathbb{Z}} \in \Omega$, we have

$$|(Tx)(n)| \leq \sum_{j=n-N}^{n} k(n,j)|f(j,x(j))| \leq M_r \max_{n\in\{1,\ldots,\omega\}} \sum_{j=1-N}^{\omega} k(n,j),$$

so $T\Omega$ is a bounded subset of $\mathcal{A}_\omega$, and $\overline{T\Omega}$ is compact. Thus, T is compact.□

PROPOSITION 3.2 *Suppose (from the listed* ASSUMPTIONS *on k, f, ψ, ξ) that (3.3), (3.4a), (3.4b), (3.4c) (3.5a), (3.5b), (3.5c). (3.6) hold and, given (3.8), suppose (where q is the function in (3.5b))*

$$\kappa_{\min}(q) > 0; \tag{3.14a}$$

there exists $M \in (0,1)$ with

$$M/\xi(M) \leq a_0\kappa_{\min}(q)/\kappa_{\max}(q); \tag{3.14b}$$

there exists $\alpha > 0$ with

$$\alpha > \kappa_{\max}(q)\psi(\alpha); \tag{3.14c}$$

and there exists $\beta > 0$, $\beta \neq \alpha$, with

$$\beta < a_0\kappa_{\min}(q)\psi(M\beta). \tag{3.14d}$$

Then, (1.2) has at least one positive periodic solution $x \in \mathcal{A}_\omega(\mathbb{Z})$, with

$$0 < \min\{\alpha,\beta\} < |x|_\omega < \max\{\alpha,\beta\};\ x(n) \geq M\min\{\alpha,\beta\} \text{ for } n \in \mathbb{Z}. \tag{3.15}$$

Proof. For $a \in \mathbb{R}$ let $\Omega_a = \{x \in \mathcal{A}_\omega : |x|_\omega < a\}$. To apply Krasnosel′skiĭ's fixed point theorem, we shall show that the following conditions hold:
(a) $T : C \to C$,
(b) $|Tx|_\omega \leq |x|_\omega$ for $x \in C \cap \partial\Omega_\alpha = S_\alpha$ and
(c) $|Tx|_\omega \geq |x|_\omega$ for $x \in C \cap \partial\Omega_\beta = S_\beta$.

(a) Let $x \in \mathcal{C}$. Then (3.5) implies that, for $n \in \{1, \ldots, \omega\}$,

$$|(Tx)(n)| \leq \psi(|x|_\omega) \max_{n \in \{1,\ldots,J\}} \sum_{j=n-N}^{n} k(n,j)q(j) = \kappa_{\max}(q)\psi(|x|_\omega). \tag{3.16}$$

On the other hand, since $x \in \mathcal{C}$, we have $x(n) \geq M|x|_\omega$ for $n \in \mathbb{Z}$, and therefore (3.5), (3.6), (3.16) and (3.14b) give for $n \in \{1, \ldots, \omega\}$,

$$\begin{aligned}(Tx)(n) &\geq a_0 \sum_{j=n-N}^{n} k(n,j)q(j)\psi(x(j)) \geq a_0\psi(M|x|_\omega) \sum_{j=n-N}^{n} k(n,j)q(j) \\ &\geq a_0\xi(M)\psi(|x|_\omega) \sum_{j=n-N}^{n} k(n,j)q(j) \geq \kappa_{\min}(q)a_0\xi(M)\psi(|x|_\omega) \\ &\geq \{\kappa_{\min}(q)/\kappa_{\max}(q)\}\, a_0\xi(M)|Tx|_\omega \geq M|Tx|_\omega;\end{aligned}$$

thus, $Tx \in \mathcal{C}$ and (a) holds.

To establish (b), let $x \in \mathcal{C} \cap \partial\Omega_\alpha = \mathcal{S}_\alpha$. In this case, $|x|_\omega = \alpha$ and $x(n) \geq M\alpha$ for all $n \in \mathbb{Z}$. Now for $n \in \{1, \ldots, \omega\}$, we have

$$|(Tx)(n)| \leq \psi(|x|_\omega) \sum_{j=n-N}^{n} k(n,j)q(j) \leq \psi(\alpha)\kappa_{\max}(q).$$

This, with (3.14c), yields $|Tx|_\omega \leq \psi(\alpha)\kappa_{\max}(q) < \alpha = |x|_\omega$; thus (b) is satisfied.

To establish (c), let $x \in \mathcal{C} \cap \partial\Omega_\beta = \mathcal{S}_\beta$. Then $|x|_\beta = \beta$ and $M\beta \leq x(n) \leq \beta$ for all $n \in \mathbb{Z}$. Now, for $n \in \{1, \ldots, \omega\}$, it follows from (3.5) that

$$|(Tx)(n)| \geq a_0 \sum_{j=n-N}^{n} k(n,j)q(j)\psi(x(j)) \geq a_0\kappa_{\min}(q)\psi(M\beta), \tag{3.17}$$

which, together with (3.14d), yields

$$(Tx)(n) \geq a_0\kappa_{\min}(q)\psi(M\beta) > \beta = |x|_\omega \text{ for } n \in \{1, \ldots, \omega\},$$

and thus $|Tx|_\omega > |x|_\omega$; that is, (c) holds.

Applying Krasnosel'skiĭ's Theorem 2.4, we conclude that (1.2) has a solution $x \in \mathcal{A}_\omega$ with $x \in \mathcal{C} \cap (\overline{\Omega_\alpha} \backslash \Omega_\beta)$ if $\beta < \alpha$, or $x \in \mathcal{C} \cap (\overline{\Omega_\beta} \backslash \Omega_\alpha)$ if $\alpha < \beta$. Finally, we note that $|x|_\omega \neq \alpha$ and $|x|_\omega \neq \beta$. In fact, if $|x|_\omega = \alpha$, then from $x = Tx$ we have $\alpha = |x|_\omega = |Tx|_\omega \leq \psi(\alpha)\kappa_{\max}(q) < \alpha = |x|_\omega$ (which is a contradiction). A similar argument shows that $|x|_\omega \neq \beta$. This completes the proof. □

Example 3.1 *Consider (compare (1.8a)) the non-linear system*

$$x(n) = \sum_{j=n-N}^{n} k(n,j)g(j)[x(j)]^\gamma \quad \textit{for } n \in \mathbb{Z}, \tag{3.18}$$

where $0 < \gamma < 1$. *Assume that* k *satisfies (3.3) and, for* $\omega \in \mathbb{N}$, *that*

$$g \in \mathcal{A}_\omega, \;\; g(n) \geq 0, \;\; \textit{and } \kappa_{\min}(g) > 0 \tag{3.19}$$

(with $\kappa_{\min}(g)$, $\kappa_{\max}(g)$ *defined by (3.8)). Then (3.18) has at least one positive periodic solution* $x \in \mathcal{A}_\omega$, *where with* $M = \frac{1}{2}\left(\kappa_{\min}(g)/\kappa_{\max}(g)\right)^{\frac{1}{1-\gamma}}$, $\beta = \frac{1}{2}M^{\frac{\gamma}{1-\gamma}}(\kappa_{\min}(g))^{\frac{1}{1-\gamma}} < |x|_\omega < 2(\kappa_{\max}(g))^{\frac{1}{1-\gamma}} = \alpha$ *and* $x(n) \geq M\beta$ *for* $n \in \mathbb{Z}$.

To see that the above result is true, we apply Proposition 3.2 with $q = g$, $f(n,u) = q(n)u^\gamma$, $\psi(u) = u^\gamma$, $a_0 = 1$ *and* $\xi(u) = u^\gamma$. *Now, the continuity and periodicity conditions on* k, f *in Proposition 3.2 and continuity and monotonicity properties of* ψ *are clearly satisfied; we verify (3.14). Now (3.14a) is satisfied by assumption. To establish (3.14b) for* ξ, *we note that*

$$\frac{M}{\xi(M)} = M^{1-\gamma} = \left(\tfrac{1}{2}\right)^{1-\gamma}\frac{\kappa_{\min}(g)}{\kappa_{\max}(g)} \leq \frac{\kappa_{\min}(g)}{\kappa_{\max}(g)} = a_0\frac{\kappa_{\min}(g)}{\kappa_{\max}(g)}.$$

Also (3.14c) holds since $\alpha/(\psi(\alpha)) = \alpha^{1-\gamma} = 2^{1-\gamma}\kappa_{\max}(g) > \kappa_{\max}(g)$. *Since* $\beta/\psi(M\beta) = \{1/M^\gamma\}\beta^{1-\gamma} = \{1/M^\gamma\}\left(\frac{1}{2}\right)^{1-\gamma}M^\gamma\kappa_{\min}(g) = \left(\frac{1}{2}\right)^{1-\gamma}\kappa_{\min}(g) < a_0\kappa_{\min}(g)$, *(3.14d) is also true. Now apply Proposition 3.2.*

With additional conditions on k and f in (1.2), applications of Proposition 3.2 will yield additional positive periodic solutions of (1.2). For completeness we provide one result on multiple solutions.

Proposition 3.3 *Suppose (from the listed* Assumptions *on* k, f, ψ, ξ*) that (3.3), (3.4a), (3.4b), (3.4c) (3.5a), (3.5b), (3.5c), (3.6) hold, and also that (3.14b) is satisfied. Also, given (3.8), suppose (where* $q \in \mathcal{A}_\omega$ *is the function in (3.5b)) that there are constants* $0 < \gamma_0 < \gamma_1 < \gamma_2$ *with (i)* $\gamma_0 < a_0\kappa_{\min}(q)\psi(M\gamma_0)$, *with (ii)* $\gamma_1 > \kappa_{\max}(q)\psi(\gamma_1)$ *and with (iii)* $\gamma_2 < a_0\kappa_{\min}(q)\psi(M\gamma_2)$. *Then (1.2) has at least two positive periodic solutions* $x_1 = \{x_1(n)\}_{n\in\mathbb{Z}}$, *and* $x_2 = \{x_2(n)\}_{n\in\mathbb{Z}} \in \mathcal{A}_\omega$ *with* $0 < \gamma_0 < |x_1|_\omega < \gamma_1 < |x_2|_\omega < \gamma_2$, $x_1(n) \geq M\gamma_0$ *and* $x_2(n) \geq M\gamma_1$ *for* $n \in \mathbb{Z}$.

Proof. The existence of x_1 follows from Proposition 3.2 with $\alpha = \gamma_1$ and $\beta = \gamma_0$, and the existence of x_2 follows from Proposition 3.2 with $\alpha = \gamma_1$ and $\beta = \gamma_2$. □

3.2 Non-negative periodic solutions via the non-linear alternative

We use the non-linear alternative to obtain an existence result for (1.2).

Proposition 3.4 *Suppose (from the listed* Assumptions *on* k, f, ψ, *and* $q \in \mathcal{A}_\omega$*) that (3.3), (3.4a), (3.4b), (3.4c), (3.5a), (3.5b) are satisfied, and in addition assume that, with* $\kappa_{\max}(q)$ *as in (3.8), there exists* $\alpha > 0$ *with*

$$\alpha > \kappa_{\max}(q)\psi(\alpha). \tag{3.20}$$

Then (1.2) has a non-negative solution $x \in \mathcal{A}_\omega$ *with* $|x|_\omega < \alpha$.

Proof. Any non-negative solution (that is, with $x(n) \geq 0$ for $n \in \mathbb{Z}$) of (1.2) is a solution of

$$x(n) = \sum_{j=n-N}^{n} k(n,j) f_\natural(j, x(j)), \quad N \in \mathbb{N}, \quad x(n) \in \mathbb{R}. \tag{3.21}$$

where

$$f_\natural(n,u) = f(n,|u|). \tag{3.22}$$

By construction, and the assumptions on f, $f_\natural(n,\cdot)$ is continuous on $\mathbb{R}$ for each $n \in \mathbb{Z}$ and

$$f_\natural(n+\omega, u) = f_\natural(n,u) \geq 0 \text{ for all } n \in \mathbb{Z} \text{ and } u \in \mathbb{R}. \tag{3.23}$$

We apply the alternative theorem to (3.21), setting

$$T_\natural x(n) = \sum_{j=n-N}^{n} k(n,j) f_\natural(j, x(j)) \tag{3.24}$$

and taking $X_{\|\cdot\|}$ to be A_ω with norm $|\cdot|_\omega$ and $U = \{x \in A_\omega \;\big|\; |x|_\omega < \alpha\}$. It is easy to see that the operator $T_\natural$ on A_ω (where $(T_\natural x)(n) = \sum_{j=n-N}^{n} k(n,j) f_\natural(j, x(j))$ for $n \in \mathbb{Z}$) maps A_ω to A_ω by conditions (3.3) and by the assumed properties of f. In addition, these properties guarantee that $T_\natural : A_\omega \to A_\omega$ is continuous and compact. Let $x_\lambda \in A_\omega$ be any solution of

$$x_\lambda(n) = \lambda\{\sum_{j=n-N}^{n} k(n,j) f_\natural(j, x_\lambda(j))\}, \quad n \in \mathbb{Z},$$

for $0 < \lambda < 1$. Notice that (3.3) and (3.4) imply $x_\lambda(n) \geq 0$ for all $n \in \mathbb{Z}$. Now for $n \in \{1,\ldots,\omega\}$, we have $|x_\lambda(n)| \leq \sum_{j=n-N}^{n} k(n,j) q(j) \psi(x_\lambda(j)) \leq \psi(|x_\lambda|_\omega) \sum_{j=n-N}^{n} k(n,j) q(j) \leq \kappa_{\max}(q) \psi(|x_\lambda|_\omega)$ and therefore

$$|x_\lambda|_\omega \leq \kappa_{\max}(q) \psi(|x_\lambda|_\omega). \tag{3.25}$$

In addition, (3.20) and (3.25) implies that $|x_\lambda|_\omega \neq \alpha$. Apply the non-linear alternative: since we have shown that option (ii) (in the statement of that theorem) cannot occur, we deduce that (1.2) has a solution $x = \{x(n)\}_{n\in\mathbb{Z}} \in \mathcal{A}_\omega$ with $x(n) \geq 0$ for all $n \in \mathbb{Z}$. Further, $|x|_\omega < \alpha$. (We have $|x|_\omega \leq \alpha$ by the non-linear alternative and $|x|_\omega \neq \alpha$ by an argument similar to that used to show that $|x_\lambda|_\omega \neq \alpha$.) □

Remark 3.2 *If there is $n_0 \in \{1,\ldots,\omega\}$ such that $k(n_0,n_0) f(n_0,0) > 0$, then the* null *periodic sequence, $\{y(n) = 0\}_{n\in\mathbb{Z}}$, is not a solution of (1.2). Then, the non-negative solution x with $|x|_\omega < \alpha$, in Proposition 3.4, satisfies $|x|_\omega > 0$.*

3.3 Non-negative periodic solutions via the Leggett-Williams theorem

It is possible to use the Leggett-Williams fixed point theorem to establish the existence of non-negative periodic solutions of (1.2).

Proposition 3.5 *Suppose (from the listed* Assumptions *on k, f, ψ, and $q \in \mathcal{A}_\omega$) that (3.3), (3.4a), (3.4b), (3.4c), (3.5) hold; suppose, further, that χ satisfies (3.7). In addition, assume that*

$$\kappa_{\min}(q) > 0; \tag{3.26}$$

that there exists $R > r > 0$ such that $\dfrac{\chi(u)}{u}$ is nonincreasing on $(0, r)$ and

$$r < \kappa_{\min}(q)\chi(r), \tag{3.27a}$$

$$R > \psi(R)\kappa_{\max}(q). \tag{3.27b}$$

Then (1.2) has a non-negative solution $x \in \mathcal{A}_\omega$ with $r \le |x|_\omega < R$.

Proof. In the Leggett-Williams theorem (Theorem 2.6), take $\mathcal{X} = \mathcal{A}_\omega$ and $C = \{x \in \mathcal{A}_\omega \big| x(n) \ge 0\}$ for $n \in \{1, \dots, \omega\}$, *i.e.*, $\{x \in \mathcal{A}_\omega \big| x(n) \ge 0$ for $n \in \mathbb{Z}\}$. Clearly, C is a cone in $\mathcal{A}_\omega$. Let $u_0 = \{u_0(n)\}_{n\in\mathbb{Z}}$ with $u_0(n) = 1$ for all $n \in \mathbb{Z}$ and note that $C(u_0) = \{x \in C :$ there exists $\lambda > 0$ with $x(n) \ge \lambda$ for $n \in \{1, \dots, \omega\}$. From (3.3) and (3.4a), it follows that $T : C \to C$ (where $(Tx)(n) = \sum_{j=n-N}^{n} k(n, j)f(j, x(j))$ for $n \in \mathbb{Z}$ and $x \in C$) is continuous and compact.

To apply Theorem 2.6, we first show

$$|Tx|_\omega \le |x|_\omega \text{ for } x \in S_R = \{x \in C \big| |x|_\omega = R\}. \tag{3.28}$$

If $x \in S_R$, then $|x|_\omega = R$ and for $n \in \{1, \dots, \omega\}$, since $\displaystyle\sum_{j=n-N}^{n} k(n, j)q(j)\psi(x(j)) \le \psi(|x|_\omega) \sum_{j=n-N}^{n} k(n, j)q(j)$, we have

$$(Tx)(n) \le \psi(R)\kappa_{\max}(q). \tag{3.29}$$

This, together with (3.27b), gives

$$|Tx|_\omega \le \psi(R)\kappa_{\max}(q) < R = |x|_\omega, \tag{3.30}$$

which implies that (3.28) is true. Next we show

$$Tx \not\le x \quad \text{for } x \in S_r \cap C(u_0), \text{ where } S_r = \{x \in C \big| |x|_\omega = r\}. \tag{3.31}$$

To show this, let $x = \{x(n)\}_{n\in\mathbb{Z}} \in S_r \cap C(u_0)$, hence $|x|_\omega = r$ and $r \ge x(n) > 0$ for $n \in \{1, \dots, \omega\}$. Now, for $n \in \{1, \dots, \omega\}$, we have

$$(Tx)(n) \ge \sum_{j=n-N}^{n} k(n, j)q(j)\frac{\chi(x(j))}{x(j)}x(j) \ge \frac{\chi(r)}{r} \sum_{j=n-N}^{n} k(n, j)q(j)x(j).$$

Let $n_0 \in \{1, \dots, \omega\}$ be such that $\min_{n\in\{1,\dots,\omega\}} x(n) = x(n_0)$ and this together with the previous inequality yields, for $n \in \{1, \dots, \omega\}$,

$$(Tx)(n) \geq \frac{\chi(r)}{r} x(n_0) \sum_{j=n-N}^{n} k(n,j)q(j) \geq \left(\frac{\chi(r)}{r}\kappa_{\min}(q)\right) x(n_0).$$

By (3.27a) we obtain $(Tx)(n) > x(n_0)$ for $n \in \{1, \dots, \omega\}$ and $(Tx)(n_0) > x(n_0)$, which means that (3.31) is true.

Applying Theorem 2.6, we conclude that (1.2) has a non-negative periodic solution $x = \{x(n)\}_{n\in\mathbb{Z}} \in C$ with $r \leq |x|_\omega \leq R$. Note that $|x|_\omega \neq R$ since if $|x|_\omega = R$, then from $x = Tx$, (3.29) and (3.30), we have $R = |x|_\omega = |Tx|_\omega \leq \psi(R)\kappa_{\max}(q) < R = |x|_\omega$, a contradiction. □

Example 3.2 *Consider the following discrete non-linear system*

$$x(n) = \sum_{j=n-N}^{n} k(n,j)g(j)([x(j)]^\alpha + [x(j)]^\beta), \quad n \in \mathbb{Z}, \tag{3.32}$$

with $0 < \alpha < 1$, $\beta \geq 1$ *and (3.3) satisfied. In addition assume*

$$g \in A_w \text{ with } g(n+\omega) = g(n) \geq 0 \text{ for all } n \in \mathbb{Z}, \tag{3.33}$$

$$\kappa_{\min}(g) > 0 \quad \text{and} \quad \kappa_{\max}(g) < \tfrac{1}{2}. \tag{3.34}$$

Then (3.32) has a non-negative solution $x \in \mathcal{A}_\omega$ *with*

$$\left(\tfrac{1}{2}\kappa_{\min}(g)\right)^{\frac{1}{1-\alpha}} \equiv \left(\frac{1}{2} \min_{n\in\{1,\dots,\omega\}} \sum_{j=n-N}^{n} k(n,j)g(j)\right)^{\frac{1}{1-\alpha}} \leq |x|_\omega < 1.$$

To establish this, let $f(n,u) = g(n)[u^\alpha + u^\beta]$, $\psi(u) = [u^\alpha + u^\beta]$, $\chi(u) = u^\alpha$, *and* $q(n) = g(n)$ *with* $r = \left(\frac{1}{2}\kappa_{\min}(g)\right)^{\frac{1}{1-\alpha}}$ *and* $R = 1$. *Obviously, (3.4a), (3.7) and (3.26) hold. To establish (3.27a), notice that* $r/\chi(r) = r^{1-\alpha} = \frac{1}{2}\kappa_{\min}(g) < \kappa_{\min}(g)$. *Also,* $\chi(u)/u = 1/u^{1-\alpha}$ *is non-increasing on* $[0, r]$ *as* $0 < \alpha < 1$, *and finally (3.27b) holds with* $R = 1$. *We can now apply Proposition 3.5.*

4 Discretization

We develop further the discussion of the discretization of (1.1). We first pause to indicate the general nature of results found concerning (1.1), in the literature and refer to the application of Krasnosel′skiĭ's fixed point theorem in [2, p. 139]. We shall require continuity of k and f. With such modifications we have the following version of [2, Theorem 4.4.1]:

Theorem 4.1 *Given (1.1) with* $0 < \tau \in \mathbb{R}_+$, *suppose (i) that* k *satisfies (3.9a) of* Assumption *5 (ii)* f *satisfies (3.9b) of* Assumption *5 and (iii)* ψ, q, m, k

and f *collectively satisfy (3.13). Suppose (iv) that (3.14) hold when* $\kappa_{\max,\min}(q)$ *are replaced by* $\widehat{\kappa}_{\max,\min}(\mathsf{q}) \equiv \widehat{\kappa}^{\tau}_{\max\min}(\mathsf{q})$ *with*

$$\widehat{\kappa}^{\tau}_{\max}(\mathsf{q}) = \inf_{t\in[0,\varpi]} \int_{t-\tau}^{t} \mathsf{k}(t,s)\mathsf{q}(s)\ \mathrm{d}s,\ \widehat{\kappa}^{\tau}_{\min}(\mathsf{q}) = \sup_{t\in[0,\varpi]} \int_{t-\tau}^{t} \mathsf{k}(t,s)\mathsf{q}(s)\ \mathrm{d}s. \quad (4.1)$$

Then there exists at least one ϖ*-periodic solution* x *of (1.1) such that*

$$0 < \min\{\alpha,\beta\} < \sup_t |\mathsf{x}(t)| < \max\{\alpha,\beta\};\ \mathsf{x}(t) \geq M\min\{\alpha,\beta\}\ \textit{for}\ t\in\mathbb{R}. \quad (4.2)$$

A ϖ-periodic solution x is a solution that satisfies $\mathsf{x}(t+\varpi) = \mathsf{x}(t)$ for all $t\in\mathbb{R}$.

Proposition 4.1 *Suppose that* $0 < \tau \in \mathbb{R}_+$ *and that* $\tilde\tau \equiv \tilde\tau(\varepsilon)$ *with* $0 < \tau-\varepsilon < \tilde\tau \leq \tau$*. If the conditions of Theorem 4.1 apply, they apply also when* τ *is replaced by* $\tilde\tau$ *provided that* ε *is sufficiently small — and the conclusion of Theorem 4.1 applies to a* ϖ*-periodic solution*

$$\tilde{\mathsf{x}}(t) = \int_{t-\tilde\tau}^{t} \mathsf{k}(t,s)\mathsf{f}(s,\tilde{\mathsf{x}}(s))\ \mathrm{d}s. \quad (4.3)$$

Further, provided that ε *is sufficiently small, if the conditions of Theorem 4.1 apply to (4.3) they establish the existence of a* ϖ*-periodic solution* x *of (1.1).*

Proof: We are concerned only with the effect of replacing $\kappa_{\max,\min}(q)$ in (3.14) by $\widehat{\kappa}^{\tilde\tau}_{\max,\min}(q)$ insead of by $\widehat{\kappa}^{\tau}_{\max,\min}(q)$. Under the assumptions, the integrals $\int_{t'}^{t} \mathsf{k}(t,s)\mathsf{q}(s)\ \mathrm{d}s$ depend continuously on $t' \in [\tau,\tau+\varepsilon]$, uniformly in t. The integral with lower limit $t-\tau$ and that with lower limit $t-\tilde\tau$ are therefore arbitrarily close (uniformly in t) for correspondingly small ε. Indeed, $|\int_{t-\tau}^{t} \mathsf{k}(t,s)\mathsf{q}(s)\ \mathrm{d}s - \int_{t-\tilde\tau}^{t} \mathsf{k}(t,s)\mathsf{q}(s)\ \mathrm{d}s| \leq \varepsilon \sup_{t\in\mathbb{R}} \sup_{s\in[t-\tau,t-\tilde\tau]} |\mathsf{k}(t,s)| \sup_{s\in[t-\tau,t-\tilde\tau]} |\mathsf{g}(s)|$ and hence is bounded by $\varepsilon \sup_{t\in[0,\varpi]} \sup_{s\in[t-\tau,t-\tilde\tau]} |\mathsf{k}(t,s)| \sup_{s\in[t-\tau,t-\tilde\tau]} |\mathsf{g}(s)|$. Thus the pair $(\widehat{\kappa}^{\tau}_{\min}(q), \widehat{\kappa}^{\tilde\tau}_{\min}(q))$ and the pair $(\widehat{\kappa}^{\tau}_{\max}(q), \widehat{\kappa}^{\tilde\tau}_{\max}(q))$ are in each case arbitrarily close for $\tau - \tilde\tau$ sufficiently close. □

We shall exploit the preceding result, which indicates (in broad terms) that one can consider $\tau > 0$ to be replaced by a nearby $\tilde\tau$. Assume that

$$\tau - \varepsilon < \tilde\tau \leq \tau,\ \text{and}\ \tilde\tau = N_{\tilde\tau}\mathfrak{h}\ \text{where}\ N_{\tilde\tau} \in \mathbb{N}, \quad (4.4)$$

for sufficiently small $\varepsilon > 0$ that Proposition 4.1 applies. The process for ensuring (4.4) is discussed later (§4.3). We consider discretization of

$$\mathsf{x}(t) = \int_{t-\tilde\tau}^{t} \mathsf{k}(t,s)\mathsf{f}(s,\mathsf{x}(s))\mathrm{d}s, \quad \text{with}\ \tilde\tau > 0,\ \mathsf{x}(t)\in\mathbb{R}, \quad (4.5)$$

(which is (1.1), with τ replaced by $\tilde\tau$) using quadrature. When we discretize (4.5), we seek (for $N_{\tilde\tau} \in \mathbb{Z}_+$, $\mathfrak{h} = \tilde\tau/N_{\tilde\tau}$) a discrete system of the type

$$\widetilde{x}(n) = \sum_{j=n-N_{\tilde\tau}}^{n} \mathfrak{h} W_{n-j} \mathsf{k}(n\mathfrak{h}, j\mathfrak{h})\mathsf{f}(j\mathfrak{h}, \widetilde{x}(j)), \quad \widetilde{x}(n)\in\mathbb{R}, \quad (4.6)$$

with $\tilde{x}(j) \approx \mathsf{x}(j\mathfrak{h})$. Our aim is to examine conditions on the integral equation (1.1) which allow the analysis, as in [1], of periodic solutions, and to discuss whether one can apply, to (4.6), the discrete analysis developed earlier.

We assume that k satisfies conditions (for example, those in Theorem 4.1) that guarantee the existence of a ϖ-periodic solution x of (1.7), on the assumption that ε is sufficiently small that the conditions continue to be satisfied when τ is replaced by $\tilde{\tau}$ in (4.4); see Proposition 4.1. To simulate Theorem 4.1 (and similar results) using discrete equations, we present (see §4.1 and B) various quadratures that can be used to discretize (1.1), via (4.5), while ensuring the existence of periodic solutions of (4.6).

Remark 4.1 *From a purely computational perspective, one might discretize (1.1) using the given τ, perhaps by implementing an error control mechanism, but without any attempt to conserve periodic equations. It is to be expected that the analysis carried out here will yield insight into approximations obtained by such means.*

4.1 Quadrature rules: basic properties and examples

We restrict attention to simple approximations associated with sampling the integrand at *equally-spaced abscissae*. Though we need to discretize integrals over $[t-\tau, t]$ (first replacing the integral by that over $[t-\tilde{\tau}, t]$), our quadrature rule is defined by the approximation for an integral over $[0, 1]$, of the type

$$\int_0^1 \psi(s)\ \mathrm{d}s \approx \sum_{\ell=0}^{N} w_\ell\, \psi(\ell\, h) =: \mathcal{Q}_{1/N}(\psi), \quad h = 1/N, \tag{4.7}$$

with the convention that an affine change of variable is used to secure, for arbitrary finite $a < b$, the related, or *induced*, approximation

$$\mathcal{Q}_{1/N}^{[a,b]}(\psi) := \sum_{j=0}^{N} \{(b-a)w_j\}\psi(a + jh_\natural) \approx \int_a^b \psi(s)\ \mathrm{d}s, \quad h_\natural = (b-a)/N \tag{4.8}$$

($\mathcal{Q}_{1/N}(\psi)$ is $\mathcal{Q}_{1/N}^{[0,1]}(\psi)$). The sum $\mathcal{Q}_{(b-a)/N}^{[a,b]}(\psi)$ in (4.8) is the *quadrature "approximation"* to $\int_a^b \psi(s)\ \mathrm{d}s$. We define the *quadrature "rule"* $\mathcal{Q}$ as the map that, with $N \in \mathbb{N}$, assigns to finite $a, b \in \mathbb{R}$ and $\psi \in R[a, b]$ the value (4.8). We restrict the *permitted* quadrature (4.7) to formulae satisfying

$$\sum_{\ell=0}^{N} w_\ell = 1, \text{ and } w_j \geq 0 \text{ for } j \in \{0, 1, \ldots, N\}. \tag{4.9}$$

Certain Newton-Cotes rules, the 2-point Radau rule, the composite versions, and classical Romberg and certain Gregory rules (see below) provide examples.

Both (4.7) and (4.8) define the quadrature rule $\mathcal{Q}$, but a rule is defined uniquely by the *"weights"* $\{w_\ell\}_0^N$ and the abscissae in (4.7). Since some weights w_j may vanish, the values of h and N are fixed by requiring that (4.7) with $h \in (0, 1]$ cannot be rewritten using a larger value $h^* > h$ (smaller value $N^* < N \in \mathbb{N}$

where $h^* = 1/N^*$), as an approximation $\sum_{\ell=0}^{N^*} w_\ell^* \,\psi(\ell\, h^*)$. A quadrature *family* $\mathfrak{Q}$ (a collection of rules) is defined by (4.7) for $h \in \mathfrak{H}^{\mathfrak{Q}} \subset (0,1]$. We assume

$$\mathfrak{H}^{\mathfrak{Q}} := \{h_1, h_2, h_3, \ldots\}, \quad \mathfrak{N}^{\mathfrak{Q}} := \{N^{[1]}, N^{[2]}, N^{[3]}, \ldots\} \; (h_\ell = 1/N^{[\ell]}), \qquad (4.10)$$

are defined by a monotonically decreasing sequence of values h_ℓ with $N^{[\ell]} \in \mathfrak{N}^{\mathfrak{Q}} \subseteq \mathbb{N}$. The choice of $h \in \mathfrak{H}^{\mathfrak{Q}}$ (equivalently $N \in \mathfrak{N}^{\mathfrak{Q}}$) is assumed to define uniquely a particular rule in $\mathfrak{Q}$. The selection of a particular type of quadrature rule restricts $\mathfrak{N}^{\mathfrak{Q}}$ (the possible $N^{[\ell]}$): for m-times repeated rules (see below) $N^{[\ell]}$ has the form $m \times N^{[0]}$ while for classical Romberg rules (see [4,5] and Appendix B) $N^{[\ell]}$ has the form $2^m \times N^{[0]}$ ($\ell, m \in \mathbb{N}$). Subsequently, we employ approximations

$$\int_{(n-N)\mathfrak{h}}^{n\mathfrak{h}} \psi(s)\; \mathrm{d}s \approx \sum_{j=n-N}^{n} \mathfrak{h} W_{n-j} \psi(j\mathfrak{h}) \quad (\text{with } n \in \mathbb{N},\; \sum_{j=0}^{N} W_j = N), \qquad (4.11)$$

derived as particular cases of (4.8) on setting $a = (n-N)\mathfrak{h}$ and $b = n\mathfrak{h}$.

We now describe some additional families of quadrature. A *composite* or *m-times repeated* quadrature formulae ($m \in \mathbb{N}$) is based on summation over ℓ of the contributions $\mathcal{Q}^{[\ell h, (\ell+1)h]}(\psi)$ (*cf.* (4.8)) to give the approximation

$$\int_0^1 \psi(s)\; \mathrm{d}s = \sum_{\ell=0}^{m-1} \int_{\ell/m}^{(\ell+1)/m} \psi(s)\; \mathrm{d}s \approx$$

$$\sum_{\ell=0}^{m-1} \mathcal{Q}_{1/mN}^{[\ell/m, (\ell+1)/m]}(\psi) =: (m \times \mathcal{Q}_{1/N})(\psi) \quad (\text{for } m \in \mathbb{N}). \qquad (4.12)$$

Approximation (4.12) is of the generic form (4.7); $(m \times \mathcal{Q}_{1/N})(\psi)$ could be written $\mathrm{Q}_{1/mN}(\psi)$. As (4.8) followed from (4.7), so one obtains, from (4.12),

$$\int_a^b \psi(s)\; \mathrm{d}s \approx (m \times \mathcal{Q}_{(b-a)/N})^{[a,b]}(\psi) = \mathrm{Q}_{(b-a)/mN}^{[a,b]}(\psi) \quad (\text{for } m \in \mathbb{N}). \qquad (4.13)$$

For Newton-Cotes rules, N is limited by (4.9), so $h = 1/N$ is bounded away from zero, but the use of composite versions with increasing m overcomes this.

We denote by $R[0,1]$ the space of *bounded Riemann-integrable functions* on $[0,1]$. If the points $\boldsymbol{\eta} = \{\eta_\ell\}_0^{M+1}$ and $\boldsymbol{\zeta} = \{\zeta_\ell\}_0^{M}$ together satisfy $0 = \eta_0 \le \zeta_1 \le \eta_1 \ldots \le \eta_M \le \zeta_M \le \eta_{M+1} = 1$ with $\zeta_\ell \ne \zeta_{\ell+1}$, $\ell \in \{0, 1, \ldots, M\}$. If $\psi \in R[0,1]$, $\Sigma(\boldsymbol{\eta}, \boldsymbol{\zeta}; \psi) = \sum_{\ell=0}^{M} \{\eta_{\ell+1} - \eta_\ell\} \psi(\zeta_\ell)$ is a *Riemann sum*. Below, Proposition 4.2 exploits a link between certain quadratures and Riemann sums. If $M \to \infty$ so that $\sup\{\eta_{\ell+1} - \eta_\ell\} \searrow 0$ then $\Sigma(\boldsymbol{\eta}, \boldsymbol{\zeta}; \psi) \to \int_0^1 \psi(s)\; \mathrm{d}s$.

4.2 Convergence of quadrature rules as $h \to 0$

The types of permitted quadrature rules discussed above provide generic sums, denoted $\{\mathcal{Q}_h(\psi)\}$, that approximate the integral $\int_0^1 \psi(s)\; \mathrm{d}s$. Thus, the expression

$$\mathcal{Q}_h^T(\psi) = \tfrac{1}{2}h\psi(0) + h\psi(h) + \cdots + h\psi(1-h) + \tfrac{1}{2}h\psi(1), \quad h = 1/m,\ N \in \mathbb{N}, \tag{4.14}$$

(*cf.* (B.1)) can be computed for $h = 1, h = \frac{1}{2}, h = \frac{1}{4}, h = \frac{1}{8}, \ldots$, or for $h = 1, \frac{1}{5}, h = \frac{1}{25}, h = \frac{1}{125}, \ldots$ and in general for $h = h_0, h = h_1, h = h_2, \ldots$ where $h_\ell = 1/N_\ell$, $N_\ell \in \mathbb{N}$, and $N_{\ell+1} > N_\ell$. The elements of the corresponding sequence $\{\mathcal{Q}_{h_\ell}^T(\psi)\}$ converge (as $h_\ell \to 0$) to a limit that coincides with the integral if $\psi \in R[0,1]$. In general, convergence can be arbitrarily slow, or as $\mathcal{O}(h_\ell^\rho)$ for arbitrary $\rho \in \mathbb{N}$ — depending on the integrand. Similar convergence issues arise for a general family of quadratures and a sequence of positive values $h \in \mathfrak{H}^{\mathfrak{Q}} = \{h_1, h_2, h_3, \ldots\}$ wherein $\lim_{\ell\to\infty} h_\ell = 0$.

Definition 4.2 *The family* $\mathfrak{Q} = \{\mathcal{Q}_h\}_{h\in\mathfrak{H}^{\mathfrak{Q}}}$ *is termed* convergent *if, for any* $\psi \in R[0,1]$, $\displaystyle\lim_{h\searrow 0,\, h\in\mathfrak{H}^{\mathfrak{Q}}} \mathcal{Q}_h^{[0,1]}(\psi) = \int_0^1 \psi(s)\ \mathrm{d}s.$

Assumption 6 *The family* $\mathfrak{Q}$ *is convergent.*

Proposition 4.2 *Repeated quadrature rules, classical Romberg rules, and Gregory rules having a fixed number of correction terms, using abscissae at step* $h \in \mathfrak{H}^{\mathfrak{Q}}$*, define families that satisfy Assumption 6.*

Proof. The composite quadrature $(m \times \mathcal{Q})$ yields a quadrature approximation that is the weighted sum of a finite set of Riemann sums; the classical Romberg approximations are Riemann sums [4, p. 123], [5, Theorem 1]; Gregory rules of the stated type differ by $\mathcal{O}(h)\,\|\psi\|_\infty$ from a Riemann sum, and the Riemann sums referred to here converge to the required integral as $h \searrow 0$. □

Proposition 4.3 *Suppose that* $\mathfrak{U}$ *is a set of functions defined on* $[0,1]$ *that are uniformly bounded and equi-continuous, and Assumption 6 is valid. Then* $\displaystyle\lim_{h\searrow 0}\sup_{u\in\mathfrak{U}} |\mathcal{Q}_h(u) - \int_0^1 u(s)\ \mathrm{d}s| = 0,$ *where* $h \searrow 0$ *with* $h \in \mathfrak{H}$.

Proof If $u \in C[0,1]$ then *a fortiori* $u \in R[0,1]$. Hence, for any $u \in C[0,1]$, $\lim_{h\searrow 0} |\mathcal{Q}_h(u) - \int_0^1 u(s)\ \mathrm{d}s| = 0$. This convergence is uniform on compact sets, and $\mathfrak{U}$ is compact in $C[0,1]$ by the Arzela-Ascoli theorem. □

Proposition 4.4 *Suppose that* $\mathsf{k}(t,s)$, $f(t,u)$ *satisfy Assumption 5 and* $\mathsf{q} \in \mathcal{A}_\varpi$*. Then the family of integrands* $\mathsf{k}(t, t+(\sigma-1)\tau)\mathsf{f}(t+(\sigma-1)\tau, \mathsf{q}(t+(\sigma-1)\tau))$ *for* $\sigma \in [0,1]$ *is uniformly bounded and equicontinuous for* $t \in \mathbb{R}$.

Proof Write $\mathsf{g}(t+(\sigma-1)\tau)) = \mathsf{f}(t+(\sigma-1)\tau, \mathsf{q}(t+(\sigma-1)\tau))$; by assumption, $\mathsf{g} \in \mathcal{A}_\varpi$ is continuous and $\mathsf{k}(t, t+(\sigma-1)\tau)\mathsf{g}(t+(\sigma-1)\tau)$ is continuous for $t \in [0,\varpi]$, $\sigma \in [0,1]$. (The case $f(t,u) = u$, $\mathsf{g} = \mathsf{q}$ is included.) Now,

$$\sup_{t\in\mathbb{R}} |\mathsf{k}(t, t+(\sigma-1)\tau)\mathsf{g}(t+(\sigma-1)\tau)| = \sup_{t\in[0,\varpi]} |\mathsf{k}(t, t+(\sigma-1)\tau)\mathsf{g}(t+(\sigma-1)\tau)|$$

is finite; furthermore

$$\sup_{t\in\mathbb{R}} |\mathsf{k}(t,t+(\sigma'-1)\tau)\mathsf{g}(t+(\sigma'-1)\tau) - \mathsf{k}(t,t+(\sigma''-1)\tau)\mathsf{g}(t+(\sigma''-1)\tau)| =$$

$$\sup_{t\in[0,\varpi]} |\mathsf{k}(t,t+(\sigma'-1)\tau)\mathsf{g}(t+(\sigma'-1)\tau) - \mathsf{k}(t,t+(\sigma''-1)\tau)\mathsf{g}(t+(\sigma''-1)\tau)|$$

which tends to 0 as $|\sigma'-\sigma''| \to 0$ by virtue of the uniform continuity of $\mathsf{k}(t,t+(\sigma-1)\tau)\mathsf{g}(t+(\sigma-1)\tau)$ for $t\in[0,\varpi]$, $\sigma\in[0,1]$. □

4.3 Approximate integration on $[t-\tau,t]$

For $h\in\mathfrak{H}^{\mathfrak{Q}}$, $\mathcal{Q}_h\in\mathfrak{Q}$ induces (with finite $a<b\in\mathbb{R}$) a corresponding formula $\mathcal{Q}_{\mathfrak{h}}^{[a,b]}(\psi)$ ($\mathfrak{h}=(b-a)h$, $\psi\in R[a,b]$), with $\lim_{\mathfrak{h}\searrow 0,\,\mathfrak{h}\in(b-a)\mathfrak{H}^{\mathfrak{Q}}} \mathcal{Q}_h^{[a,b]}(\psi)=\int_a^b\psi(s)\,\mathrm{d}s$. (The notation $(b-a)\mathfrak{H}^{\mathfrak{Q}}$ denotes $\{(b-a)h_1,(b-a)h_2,(b-a)h_3,\ldots\}$.) Henceforth, we are mainly concerned with integrands $\psi(s)$ of the form $\mathsf{k}(t,s)\mathsf{f}(s,\mathsf{x}(s))$ with $t\in\mathbb{R}$ (integrated for $s\in[t-\tau,t]$). Given arbitrary $\tau>0$ and $t\in\mathbb{R}$, a convergent family of quadrature rules $\mathfrak{Q}$ induces corresponding $\{\mathcal{Q}_{\mathfrak{h}}^{[t-\tau,t]}\}$ that generate discrete equations obtained from (1.1). For the specific τ in (1.1), we require quadrature (*cf.* (4.11)) to approximate integrals over $[t-\tau,t]$, in which $t=n\mathfrak{h}$, $\mathfrak{h}\in\mathfrak{H}_\varpi\cap\mathfrak{H}_\tau$. If $\tau/\varpi\notin\mathbb{Q}_+$, this cannot be achieved, so we replace τ by an approximation $\tilde\tau$ with $\tilde\tau/\varpi\in\mathbb{Q}_+$, and $\mathfrak{h}\in\mathfrak{H}_\varpi\cap\mathfrak{H}_{\tilde\tau}$. The approximation $\tilde\tau\approx\tau$ (with $\tilde\tau\le\tau$) can be made arbitrarily close by taking suitable sufficiently small $\mathfrak{h}\in\mathfrak{H}_\varpi$. To use rules from the family $\mathfrak{Q}$ we also require $\mathfrak{h}\in\widetilde\tau\mathfrak{H}^{\mathfrak{Q}}$; this is a refinement of (4.4) requiring $N_{\widetilde\tau}\in\mathbb{N}$; now, $N_{\widetilde\tau}\in\mathfrak{N}^{\mathfrak{Q}}$. In commonplace families of quadrature, the members $N^{[\ell]}$ in the sequence $\mathfrak{N}^{\mathfrak{Q}}$ form either a increasing arithmetic progression or an increasing geometric progression.

Assumption 7 *We are given positive, τ,ϖ, and a family of quadrature rules $\{\mathfrak{Q}_h\}_{h\in\mathfrak{H}^{\mathfrak{Q}}}$ associated with $\mathfrak{N}^{\mathfrak{Q}}$ in (4.10). We assume that, given arbitrary $\varepsilon>0$, there exists a corresponding $\mathfrak{h}^\star>0$ such that, whenever $\mathfrak{h}\le\mathfrak{h}_\star$ and $\mathfrak{h}\in\mathfrak{H}_\varpi$, there exists $\widetilde\tau\in[\tau-\varepsilon,\tau]$ with $\widetilde\tau=N_{\widetilde\tau}\mathfrak{h}$, where $N_{\widetilde\tau}\in\mathfrak{N}^{\mathfrak{Q}}$.*

Remark 4.2 *In Assumption 7, $\mathfrak{h}\in\mathfrak{H}_{\widetilde\tau}\cap\mathfrak{H}_\varpi\cap\widetilde\tau\mathfrak{H}_{\mathfrak{Q}}$ (and this condition is to be satisfied by $\mathfrak{h}^\star$, in the sense that $\mathfrak{h}^\star\in\mathfrak{H}_{\widetilde\tau^\star}\cap\mathfrak{H}_\varpi\cap\widetilde\tau^\star\mathfrak{H}_{\mathfrak{Q}}$). Suppose $\tau/\varpi=\varrho\in\mathbb{R}$; if $\widetilde\tau/\varpi\in\mathbb{Q}_+$ then $\widetilde\tau/\varpi=\mu_1/\mu_2$ with $\mu_{1,2}\in\mathbb{N}$. If $\mathfrak{h}\in\widetilde\tau\mathfrak{H}^{\mathfrak{Q}}$ then $\mathfrak{h}=\widetilde\tau/\widetilde N$ where $\widetilde N\in\mathfrak{N}^{\mathfrak{Q}}$ (then $\widetilde N=\widetilde\tau/\mathfrak{h}$). If $\mathfrak{h}\in\mathfrak{H}_\varpi$; then $N_\varpi:=\varpi/\mathfrak{h}\in\mathbb{N}$. Hence, $\mu_1/\mu_2=\widetilde N/N_\varpi$ where $\widetilde N\in\mathfrak{N}^{\mathfrak{Q}}$ and we need to be able to approximate ϱ arbitrarily closely by fractions of the form $\widetilde N/N_\varpi$.*

Given ϖ, $\mathfrak{h}$, with $\varpi=N_\varpi\mathfrak{h}$, identify $\omega=\omega(\mathfrak{h})$ with N_ϖ, and write $\omega^\star=\omega(\mathfrak{h}^\star)$ for $N_\varpi^\star$; if $\mathfrak{h}<\mathfrak{h}^\star$ then $N_\varpi>N_\varpi^\star$. Select $N_{\tilde\tau}$ so that $N_{\tilde\tau}=\tilde\tau/\mathfrak{h}\in\mathfrak{N}^{\mathfrak{Q}}$ where

$$\tilde\tau=\frac{N_{\tilde\tau}}{N_\varpi}\varpi=\max_{n\in\mathfrak{N}^{\mathfrak{Q}}}\Big\{\frac{n}{N_\varpi}\varpi<\tau\Big\}\text{ and }\tilde\tau^\star=\frac{N_{\tilde\tau^\star}}{N_\varpi^\star}\varpi=\max_{n^\star\in\mathfrak{N}^{\mathfrak{Q}}}\Big\{\frac{n^\star}{N_\varpi^\star}\varpi<\tau\Big\}.$$

To ensure that $0\le\tau-\widetilde\tau<\tau-\widetilde\tau^\star$ when $\mathfrak{h}<\mathfrak{h}^\star$, we require $N_{\tilde\tau}>N_{\tilde\tau^\star}\in\mathfrak{N}^{\mathfrak{Q}}$. With $\varrho=\tau/\varpi$, $N_{\tilde\tau}$ must be the largest integer in $\mathfrak{N}^{\mathfrak{Q}}$ such that $N_{\tilde\tau}<\varrho N_\varpi$. The commonplace $\mathfrak{N}^{\mathfrak{Q}}$ give $N^{[\ell]} = \mu + N^{[\ell-1]}$ with $0 < \mu \in \mathbb{N}$ or $N^{[\ell]} = \nu \times N^{[\ell-1]}$ with $1 < \nu \in \mathbb{N}$, and, for simplicity and convenience, we take the latter case with $N^{[\ell]} = \nu^{\ell}$. Then, $N_{\tilde{\tau}} = \nu^{\tilde{\lambda}}$ with $\tilde{\lambda} = \lfloor \log_{\nu}(\varrho N_{\varpi}) \rfloor$ (and $N_{\tilde{\tau}} > N_{\tilde{\tau}^{\star}}$).

Our approximate integration proceeds, with a choice of $\mathfrak{h} \in \mathfrak{H}_{\varpi} \subset (0, \tau]$, to determination of $\tilde{\tau}$, to application of a quadrature rule. When

$$|\int_{t-\tau}^{t} \psi(s)\ \mathrm{d}s - \int_{t-\tilde{\tau}}^{t} \psi(s)\ \mathrm{d}s| \le \varepsilon \sup_{s \in [t-\tau, t-\tilde{\tau}]} |\psi(s)|, \tag{4.15}$$

the quadrature involved in (4.7) induces, for $t = n\mathfrak{h}$, and on setting N equal to $N_{\tilde{\tau}}$, the approximation

$$\int_{t-\tilde{\tau}}^{t} \psi(s)\ \mathrm{d}s \approx \sum_{\ell=0}^{N_{\tilde{\tau}}} \tilde{\tau} w_{\ell}\, \psi(t - \tilde{\tau} + \ell\, \mathfrak{h}) =: \mathcal{Q}_{\mathfrak{h}}^{[t-\tilde{\tau}, t]}(\psi), \quad (\mathfrak{h} = \tilde{\tau}/N_{\tilde{\tau}}), \tag{4.16}$$

so the sum in (4.16) approximates the first integral in (4.15) with a sum of the type in (4.11).

Indeed, for the integral in (4.5) over $[n\mathfrak{h} - \tilde{\tau}, n\mathfrak{h}]$, the m-times repeated quadrature corresponding to (4.7) induces the approximation

$$\int_{t-\tilde{\tau}}^{t} \psi(s)\ \mathrm{d}s \approx \sum_{j=n-N_{\tilde{\tau}}}^{n} \mathfrak{h} W_{n-j}^{(m\times\mathcal{Q})}\, \psi(j\mathfrak{h})\ (t = n\mathfrak{h}, \mathfrak{h} = \tilde{\tau}/N_{\tilde{\tau}},\ N_{\tilde{\tau}} = mN \in \mathbb{N}),$$

with $W_j^{(m\times\mathcal{Q})} = w_s$ if $j \equiv s \bmod (m)$ and $s \neq 0$, $W_0^{(m\times\mathcal{Q})} = w_N$, $W_{rm}^{(m\times\mathcal{Q})} = w_0 + w_N$ if $rm \notin \{0, N_{\tilde{\tau}}\}$, $W_{N_{\tilde{\tau}}}^{(m\times\mathcal{Q})} = w_0$. By assumption, $w_j \ge 0$ for $j \in \{0, 1, \ldots, N\}$; hence, $W_j^{(m\times\mathcal{Q})}$ is non-negative for $j \in \{0, 1, \ldots, N_{\tilde{\tau}}\}$. Likewise, a given rule R from amongst the classical Romberg rules gives an approximation expressible as $\int_0^1 \psi(s)\ \mathrm{d}s \approx \sum_{j=0}^{2^m} w_j^{[R]} \psi(j/\{2^m\})$, where $m \in \mathbb{N}$, and where $w_j^{[R]} = w_{2^m-j}^{[R]} > 0$. We thus obtain, for $t = n\mathfrak{h}$, formulae of the type

$$\int_{t-\tilde{\tau}}^{t} \psi(s)\ \mathrm{d}s \approx \sum_{j=n-2^m}^{n} \mathfrak{h} W_{n-j}^{[R]} \psi(t - \tilde{\tau} + j\mathfrak{h}),\ \mathfrak{h} = \tilde{\tau}/2^m,\ \text{with } N_{\tilde{\tau}} = 2^m. \tag{4.18}$$

(Note the restricted form of $\mathfrak{h}$.) Further, the r-th Gregory rule $G_r^{N_{\tilde{\tau}}}$ (using $N_{\tilde{\tau}} + 1$ abscissae, and with $r \le N_{\tilde{\tau}}$) gives for $t = n\mathfrak{h}$ an approximation

$$\int_{t-\tilde{\tau}}^{t} \psi(s)\ \mathrm{d}s \approx \sum_{j=n-N_{\tilde{\tau}}}^{n} \mathfrak{h} W_{n-j}^{[G]}\, \psi(j\mathfrak{h}) \quad (t = n\mathfrak{h}, \tilde{\tau} = N_{\tilde{\tau}}\mathfrak{h},\ G \equiv G_r^{N_{\tilde{\tau}}}) \tag{4.19}$$

in which $W_{n-j}^{[G]} = W_j^{[G]}$, for $j \in \{0, 1, \ldots, N_{\tilde{\tau}}\}$. These rules include the Newton-Cotes case (for $r = N_{\tilde{\tau}}$), so the weights need not be positive, as we require.

We can now address the issue of guaranteeing conditions for periodic solutions when we discretize an integral equation (1.1) for which a result like Theorem 4.1 holds. Given a ϖ-periodic solution $\mathsf{x}(t)$ of (1.1), the sequence $\{x(n) = \mathsf{x}(n\mathfrak{h})\}_{n\in\mathbb{N}}$ is ω-periodic where $\omega = \varpi/\mathfrak{h}$ and we seek conditions ensuring a ω-periodic solution of our discretized equations. Obviously, ω is a function of $\mathfrak{h}$, $\omega = \omega(\mathfrak{h})$. We now consider an equation of the type (4.6) but in the form

$$\widetilde{x}(n) = \sum_{j=n-N_{\tilde{\tau}}(\mathfrak{h})}^{n} \mathfrak{h} W_{n-j}\mathsf{k}(n\mathfrak{h}, j\mathfrak{h})\mathsf{f}(j\mathfrak{h}, \widetilde{x}(j)), \quad \mathfrak{h} \in \mathfrak{H}_\varpi, \quad \widetilde{x}(n) \in \mathbb{R}. \tag{4.20}$$

This is of the form (1.2) with $k(n,j) := \mathfrak{h}W_{n-j}\mathsf{k}(n\mathfrak{h}, j\mathfrak{h})$, $f(j,u) := \mathsf{f}(j\mathfrak{h}, u)$. We define (4.20) to be the $\mathcal{Q}_\mathfrak{h}$-based discretization of (1.1) and we examine the assumptions made in Propositions 3.1–3.5 when applied to (4.20). Now if $\varpi = \omega\mathfrak{h}$ for $\omega \in \mathbb{N}$, then $k(n,j) = k(n+\omega, j+\omega)$ when $\mathsf{k}(n\mathfrak{h}+\varpi, j\mathfrak{h}+\varpi) = \mathsf{k}(n\mathfrak{h}, jh)$, and $f(n,u) = f(n+\omega, u)$ when $\mathsf{f}(n\mathfrak{h}+\varpi, u) = \mathsf{f}(n\mathfrak{h}, u)$. The following is typical of the results we seek.

Proposition 4.5 *Given $\mathsf{q} \in \mathcal{A}_\varpi$, and f satisfying (3.9b), suppose that $\varpi = \omega\mathfrak{h}$ and define $q(n) = \mathsf{q}(n\mathfrak{h})$, $f(n,u) = \mathsf{f}(n\mathfrak{h}, u)$. (a) Suppose that ψ satisfies (3.5a) and the functions f, ψ and q satisfy $\mathsf{f}(t,u) \le \mathsf{q}(n)\psi(u)$ for all $u \in \mathbb{R}_+$ and $t \in \mathbb{R}$, Then (3.5b) is satisfied. (b) If there exists a constant $a_0 \in (0,1)$, such that f, ψ, and q satisfy $a_0\mathsf{q}(t)\psi(u) \le \mathsf{f}(t,u)$ for all $u \in \mathbb{R}_+$ and $t \in \mathbb{R}$, then (3.5c) is satisfied.*

Other assumptions in the discrete case depend on $\kappa_{\min,\max}(q)$ ($q \in \mathcal{A}_\omega$, *cf.* (3.8)), which, with $q(n) = \mathsf{q}(n\mathfrak{h})$ compare with $\widehat{\kappa}^{\widetilde{\tau}}_{\min,\max}(\mathsf{q})$ in (4.1). Since

$$\int_{t-\tau}^{t} \mathsf{k}(t,s)\mathsf{q}(s)\,\mathrm{d}s = \int_0^1 \mathsf{k}(t, t+(\sigma-1)\tau)\mathsf{q}(t+(\sigma-1)\tau)\,\mathrm{d}\sigma$$
$$\approx \sum_j w_j \mathsf{k}(t, t+(j/N_\tau - 1)\tau)\mathsf{q}(t+(j/N_\tau - 1)\tau),$$

Proposition 4.3 and Proposition 4.4 have obvious corollaries that combine (on setting $f(t,u) = u$) to give us Proposition 4.6 below. (Recall that the family $\mathfrak{Q}$ of quadrature rules inducing $\{\mathcal{Q}_\mathfrak{h}\}$ is assumed to be convergent.)

Proposition 4.6 *Suppose that $\mathsf{k}(t,s)$ satisfies (3.9a) in Assumption 5 and $\mathsf{q} \in \mathcal{A}_\varpi$ is continuous. Then, $\sum_{j=n-N_{\tilde{\tau}}(\mathfrak{h})}^{n} \mathfrak{h}W_{n-j}\mathsf{k}(t, jh)\mathsf{q}(jh) - \int_{t-\tau}^{t} \mathsf{k}(t,s)\mathsf{q}(s)\,\mathrm{d}s$ is arbitrarily small (uniformly for $t = n\mathfrak{h}$ in $\mathbb{R}$) and $\widehat{\kappa}^{\widetilde{\tau}}_{\min,\max}(\mathsf{q})$ differ from $\kappa_{\min,\max}(q)$ by arbitrarily small amounts for sufficiently small $\mathfrak{h}$.*

It now follows that our discrete theory is applicable to the equations obtained by application of convergent quadrature of the type discussed above, given sufficient conditions on the integral equation. The following result is typical:

Proposition 4.7 *Suppose the conditions of Theorem 4.1 hold, and consider the $\mathcal{Q}_\mathfrak{h}$-discretization (4.20). Then the conditions of Proposition 3.2 apply to*

(4.20) for all sufficiently small $\mathfrak{h} \in \mathfrak{H}$.

5 Questions not addressed above

Remark 4.1 indicated a route for further research. Appendix C gives an informal indication of a number of further questions not addressed above.

References

[1] R.P. Agarwal, *Difference Equations and Inequalities*, 2nd Ed., Dekker, 2000.

[2] R.P. Agarwal, M. Meehan and D. O'Regan, *Non-Linear Integral Equations and Inclusions*, Nova Science Pub., Huntington, New York, 2001.

[3] R.P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, 2001.

[4] C.T.H. Baker, *The Numerical Treatment of Integral Equations*, Clarendon Press, Oxford, 1977.

[5] C.T.H. Baker, On the nature of certain quadrature formulae and their errors, SIAM Journal on Numerical Analysis **5**: 783–804 (1968).

[6] C.T.H. Baker, G.S. Hodgson, Asymptotic expansions for integration formulas in one or more dimensions, SIAM J. Numer. Anal., **8**: 473-480 (1971).

[7] C.T.H. Baker, Y. Song, *Periodic solutions of discrete Volterra equations*, Mathematics and Computer in Simulation, **64**: 521-542 (2004).

[8] H. Brunner, *The numerical analysis of functional integral and integro-differential equations of Volterra type* Acta Numer. 13: 55–145 (2004).

[9] K.L. Cooke and J.L. Kaplan, *A periodicity threshold theorem for epidemics and population growth*, Mat. Biosc., **31**: 87-104 (1976).

[10] F. Dannan, S.N. Elaydi and P. Liu *Periodic solutions of difference equations*, J. Differ. Equations Appl. **6**: 203–232 (2000).

[11] S.N. Elaydi, S. Zhang, *Stability and periodicity of difference equations with finite delay*, Funkc. Ekvac., **37**: 401-413 (1994).

[12] T. Furumochi, *Periodic solutions of Volterra difference equations and attractivity*, Nonlinear Analysis, **47**: 4013-4024 (2001).

[13] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Inc., Boston,1988.

[14] M.A. Krasnosel′skiĭ, Positive solutions of Operator Equations, Noordhoff, Groningen, 1964 (translated from the Russian).

[15] R.W. Leggett, L.R. Williams, *A fixed point theorem with application to an infectious disease model*, J. Math. Anal. Appl., **76**: 91-97 (1980).

[16] L.A. Liusternik, V.J. Sobolev, Elements of Functional Analysis, Frederick Ungar, New York, 1961 (translated from the Russian).

[17] Y. Song, *Periodic solutions of nonlinear discrete systems with finite delay*, unpublished report, 2006.

[18] Y. Song and C.T.H. Baker, *Perturbation theory for discrete Volterra equations*, J. Differ. Equations Appl. **9**: 969-987 (2003).

A Cones and compression operators

Definition A.1 *Let $\mathcal{X} = \{X, \|\cdot\|\}$ be a Banach space (a complete, normed, linear space). (i) A closed, convex, set C in $\mathcal{X}$ is a* (positive) cone *when:*

$$(a) \quad \text{if } u \in C \text{ then } \quad \lambda u \in C \text{ for } \lambda \geq 0 \tag{A.1}$$
$$(b) \quad \text{if } u \in C \text{ and } -u \in C \text{ then } u = 0. \tag{A.2}$$

(ii) A cone C in $\mathcal{X}$ induces a partial ordering $\leqslant$ *in $\mathcal{X}$ by the definition $u \leqslant v$ if and only if $v - u \in C$. For u, v in C such that $v - u \notin C$ we write $u \nleqslant v$. A Banach space with a partial ordering induced by a cone is a* partially ordered Banach space. *(iii) An operator T is a* compression *of a cone C in an ordered Banach space if (a) $T(0) = 0$; (b) there exist r, R with $0 < r < R$ such that $T(u) \nleqslant u$ if $u \in C$, $\|u\| \leq r$, and $u \neq 0$; and also, (c) for all $\varepsilon > 0$, $(1+\varepsilon)u \nleqslant T(u)$ if $u \in C$, $\|u\| \geq R$.*

The "compression of the cone" theorem [14, p. 137] due to Krasnosel′skiĭ (1920–1997) reads as follows.

Theorem A.2 *Let the positive completely continuous operator T be a compression of the cone C. Then T has at least one non-zero fixed point on C.*

The proof of this theorem relies upon a result due to the Polish mathematician Julius Schauder (1899-1943): *A completely continuous operator that transforms a bounded convex and closed set into itself has at least one fixed point in the given set.* Theorem A.2 was refined by Leggett and Williams [15] who proved the following result, in which the conditions associated with a compression are relaxed.

Theorem A.3 *Given a cone $C \in \mathcal{X}$, define $C_\rho := \{v \in C | \|v\| \leq \rho\}$ ($\rho \in (0, \infty)$) and $C_\infty = C$. Suppose $u \in C \backslash \{0\}$ and $C[u] := \{v \in C | \alpha v \geqslant u$ for some $\alpha > 0\}$. For some $R > 0$, suppose that $T : C_R \to C$ is completely continuous with $T(0) = 0$, and there exists r with $0 < r < R$ such that $T(u) \nleqslant u$ if $u \in C$, $\|u\| = r$ and for all $\varepsilon > 0$, $(1+\varepsilon)u \nleqslant T(u)$ if $u \in C$, $\|u\| = R$. Then T has a fixed point $x \in C$ with $r \leq \|x\| \leq R$*

B Quadrature examples

In this appendix, we describe some families of quadrature with origins in composite rules. For the trapezium rule, $\mathcal{Q}^T(\psi)$ (say), where $\mathcal{Q}^T(\psi) = \frac{1}{2}\{\psi(0) + \psi(1)\}$, the composite version $(m \times \mathcal{Q}^T)(\psi)$ yields $\displaystyle\int_0^1 \psi(s)\, \mathrm{d}s \approx \sum_{j=0}^{m}{}'' h\,\psi(jh)$ (for $h = 1/m$, $m \in \mathbb{N}$). The notation $\sum''$ is commonplace in numerical analysis but here it is convenient, for the description of Romberg rules, to write $\displaystyle\sum_{j=0}^{m}{}'' h\,\psi(jh)$ as

$$T_0(h, \psi) \equiv \frac{h}{2}\psi(0) + h\psi(h) + \cdots + h\psi(1-h) + \frac{h}{2}\psi(1). \tag{B.1}$$

Classical Romberg quadrature is based on the elimination of derivative correction terms to $T_0(h, \psi)$ (see [4,6] for the rigorous details). For sufficiently smooth integrands, the *Euler-Maclaurin formula* gives, in the notation of (B.1), $\int_0^1 \psi(s)\, \mathrm{d}s =$

$T_0(h;\psi) + \alpha_{0,1}h^2\{\psi'(1) - \psi'(0)\} + \alpha_{0,2}h^4\{\psi'''(1) - \psi'''(0)\} + \cdots$. It follows, with $T_1(h/2;\psi) = \{4T_0(h/2;\psi) - T_0(h;\psi)\}/3$, that $\int_0^1 \psi(s)\ \mathrm{d}s = T_1(h/2;\psi) + \alpha_{1,2}h^4\{\psi'''(1) - \psi'''(0)\} + \cdots$. Proceeding in a like manner, one constructs further classical Romberg quadrature approximations, $T_2(h/4;\psi) := \{16T_1(h/4;\psi) - T_1(h/2;\psi)\}/15$, *etc.* Each expression $T_0(h;\psi)$, $T_1(h/2;\psi)$, $\ldots$, $T_k(h/(2^k);\psi)$, is a weighted sum of values of the integrand, as in (4.7) but with abscissae separated at a step h, $h/2$, $\ldots$, $h/(2^k)$ (respectively).

The classical Romberg scheme can be generalized (for the theoretical basis, see [6]). Another set of quadrature rules derives from *Gregory's correction terms for the repeated trapezium rule* (B.1). Gregory's formulae provide correction terms normally expressed in terms of forward differences at the left end-point of the interval of integration and backward differences at the right end-point. The correction terms [4, page 117–118] yield the successive *Gregory rules*; see Example B.2. The Gregory rules can also be generalized. We now give examples of repeated, Romberg, and Gregory quadrature over $[t - \tilde{\tau}, t]$.

Example B.1 *The repeated trapezium rule (cf. (B.1)) induces*

$$\int_{t-\tilde{\tau}}^{t} \psi(s)\ \mathrm{d}s \approx \tfrac{1}{2}\mathfrak{h}\psi(t-\tilde{\tau}) + \mathfrak{h}\sum_{j=1}^{N_{\tilde{\tau}}-1} \psi(t-\tilde{\tau}+j\mathfrak{h}) + \tfrac{1}{2}\mathfrak{h}\psi(t), \ \textit{with}\ \mathfrak{h} = \tilde{\tau}/N_{\tilde{\tau}}. \quad \text{(B.2)}$$

Simpson's rule is a 3-point Newon-Cotes rule with degree of precision 3. The composite or m-times repeated version gives $\int_{t-\tilde{\tau}}^{t} \psi(s)\ \mathrm{d}s \approx \dfrac{\mathfrak{h}}{3}\Big\{\psi(t-\tilde{\tau}) + 4\psi(t-\tilde{\tau}+\mathfrak{h}) + 2\psi(t-\tilde{\tau}+2\mathfrak{h}) + 4\psi(t-\tilde{\tau}+3\mathfrak{h}) + \cdots + 2\psi(t-2\mathfrak{h}) + 4\psi(t-\mathfrak{h}) + \psi(t)\Big\}$ *with* $2m-1$ *terms in the sum (for* $\mathfrak{h} = \tilde{\tau}/(2m)$*). (This approximation can be derived from the classical Romberg scheme.)*

Example B.2 *The successive* quadrature weights *for the repeated trapezium rule approximation in (B.2) can be displayed as*

$$\{\ \tfrac{1}{2}\mathfrak{h},\ \underbrace{\mathfrak{h},\ \mathfrak{h},\ \mathfrak{h},\ \ldots,\ \mathfrak{h},\ \mathfrak{h},\ \mathfrak{h},}_{N_{\tilde{\tau}}-1\ \textit{terms}}\ \tfrac{1}{2}h\ \}. \quad \text{(B.3a)}$$

(In the illustrations here, $N_{\tilde{\tau}} > 8$.) The first Gregory correction terms give, instead of (B.3a), the weights

$$\{\ \tfrac{5}{12}\mathfrak{h},\ \tfrac{13}{12}\mathfrak{h},\ \underbrace{\mathfrak{h},\ \mathfrak{h},\ldots,\ \mathfrak{h},\ h,}_{N_{\tilde{\tau}}-3\ \textit{terms}}\ \tfrac{13}{12}\mathfrak{h},\ \tfrac{5}{12}h\}; \quad \text{(B.3b)}$$

the first and second Gregory correction terms together provide weights

$$\{\ \tfrac{9}{24}\mathfrak{h},\ \tfrac{28}{24}\mathfrak{h},\ \tfrac{23}{24}\mathfrak{h},\ \underbrace{\mathfrak{h},\ \ldots,\ \mathfrak{h}}_{N_{\tilde{\tau}}-5\ \textit{terms}},\ \tfrac{23}{24}\mathfrak{h},\ \tfrac{28}{24}\mathfrak{h},\ \tfrac{9}{24}\mathfrak{h}\}. \quad \text{(B.3c)}$$

C Questions not addressed in the paper

The following list provides an indication of subjects for further study that have not already been mentioned in the text.

- How do the dynamics of a solution $x(\varphi;t)$ of (1.2) for $n \geq 0$ (given $x(n) = \varphi(n)$ for $n \in \{-N, 1-N \cdots, -1\}$) relate to the existence of periodic solutions of (1.2) for $t \in \mathbb{R}$? Does this emulate the way the dynamics of a solution $\mathsf{x}(\boldsymbol{\phi};t)$ of (1.1) for $t \geq 0$, given $\mathsf{x}(t) = \boldsymbol{\phi}(t)$ for $t \in [-\tau, 0]$, relate to the existence of periodic solutions x of (1.1), considered for all $t \in \mathbb{R}$?

- What is the perturbation in a solution $\mathsf{x}(t)$ if $\tau \in \mathbb{R}$ is replaced by a neighbouring $\widetilde{\tau}$? (One may formulate an equation for the sensitivity of x to τ.)

- What can be said about $\mathsf{x}(\boldsymbol{\phi}; n\mathfrak{h}) - \widetilde{x}(\varphi; n)$ for small $\mathfrak{h}$ when $\varphi(n) = \boldsymbol{\phi}(n\mathfrak{h})$? (This is easily investigated when $\mathsf{f}(t,u)$ satisfies a uniform Lipschitz condition in u.)

- Where a solution $\{x(n)\}$ of (1.2) exists, and we set $y(n) = f(n, x(n))$, we have $y(n) = f(n, \sum_{j=n-N}^{n} k(n,j)y(j))$ (for $n \in \mathbb{Z}$). These equations yield a viewpoint that may be pursued further.

- When $c = \{c(n)\}$ is suitably periodic, we might consider the equation $x(n) = \sum_{j=n-N}^{n} k(n,j)f(j, x(j)) + c(n)$, by a study of $\hat{x}(n) = x(n) - c(n)$ where $\hat{x}(n) = \sum_{j=n-N}^{n} k(n,j)f\big(j, \hat{x}(j) - c(j)\big)$.